

HAUSDORFF THEORY OF DUAL APPROXIMATION ON PLANAR CURVES

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ABSTRACT. Ten years ago, Beresnevich-Dickinson-Velani [10] initiated a project that develops the general Hausdorff measure theory of dual approximation on non-degenerate manifolds. In particular, they established the divergence part of the theory based on their general ubiquity framework. However, the convergence counterpart of the project remains wide open and represents a major challenging question in the subject. Until recently, it was not even known for any single non-degenerate manifold. In this paper, we settle this problem for all curves in \mathbb{R}^2 , which represents the first result of its kind for a general class of manifolds.

1. INTRODUCTION

1.1. Motivation and history. Let $\mathbf{q} = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\|\cdot\|$ be the distance to the nearest integer and $|\cdot|$ be the L^∞ norm of vectors in \mathbb{R}^n . In the theory of Diophantine approximation, one considers the inequality

$$\|\mathbf{q} \cdot \mathbf{x}\| < \frac{1}{|\mathbf{q}|^\nu}, \quad (1)$$

where $\mathbf{q} \cdot \mathbf{x} = q_1 x_1 + \dots + q_n x_n$ is the standard inner product on \mathbb{R}^n . In contrast to the *simultaneous approximation*, traditionally this setting (1) is referred as the *dual approximation* (see [18] for more basics).

Dirichlet's theorem asserts that if $\nu \leq n$, then for all $x \in \mathbb{R}^n$, the inequality (1) admits infinitely many solutions in \mathbf{q} . The point \mathbf{x} is called *very well approximable* (abbr. VWA) if for some $\nu > n$, (1) holds for infinitely many $\mathbf{q} \in \mathbb{Z}^n$. Thus, for VWA points, the exponent in (1) can be improved beyond Dirichlet. Nevertheless, a straightforward application of the Borel-Cantelli lemma reveals that almost all $\mathbf{x} \in \mathbb{R}^n$

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are not VWA. Thus the Dirichlet exponent $\nu = n$ is a natural threshold for (1) from the metrical point of view.

There are naturally two manners of generalization. One may ask what is the precise threshold for (1) to have infinitely many solutions when restricting \mathbf{x} to lying on a proper submanifold of \mathbb{R}^n and/or replacing the right side of (1) with a general approximation function $\psi(|\mathbf{q}|)$. This turns out to be the key question that motivates many modern developments of the theory, on which we will do a brief survey.

The theory of Diophantine approximation on manifolds dates back to 1930s when Mahler [27] raised his conjecture in transcendence theory, which states that almost all points on the Veronese curve

$$\mathcal{V}_n := \{(x, x^2, \dots, x^n) | x \in \mathbb{R}\}$$

are not VWA. This rather difficult question was eventually solved by Sprindžuk [30] in 1964. After this, Sprindžuk proposed an important general conjecture.

Conjecture 1 (Sprindžuk). Any analytic non-degenerate submanifold of \mathbb{R}^n is extremal.

A manifold \mathcal{M} is *extremal* if almost all points of \mathcal{M} (with respect to the natural induced measure on \mathcal{M}) are not VWA. A manifold \mathcal{M} is *non-degenerate* if at almost all points $\mathbf{x} \in \mathcal{M}$, \mathcal{M} has at most finite order of contact with any hyperplane that passes through \mathbf{x} . It is easily seen that an analytic manifold is non-degenerate if and only if, around almost all points, the local chart functions together with 1 are linearly independent over \mathbb{R} . Essentially, non-degeneracy guarantees that the manifold is sufficiently curved so as to deviate from any hyperplanes.

The philosophy underlying this conjecture can be summarized as that, in the sense of Diophantine approximation, a generic point on a generic submanifold of \mathbb{R}^n should behave like a generic point in \mathbb{R}^n . In other words, as far as certain Diophantine approximation properties are concerned, one cannot distinguish a generic point on a non-degenerate submanifold of \mathbb{R}^n from a generic point in \mathbb{R}^n . We will see that this very same idea arises in most of the major (solved or unsolved) questions in the area of metric Diophantine approximation on manifolds.

The fundamental Sprindžuk conjecture was proved by Kleinbock and Margulis [26] in 1998 using dynamical tools based on unipotent flows on homogeneous spaces. Their proof has served as a catalyst for many more recent developments (They actually proved the stronger Baker-Sprindžuk conjecture). Among other things, the Khintchine-Groshev theory is a natural generalization of the theory of extremal manifolds

obtained by replacing the right side of (1) with a monotonic function ψ of $|\mathbf{q}|$.

From now on, we will call the decreasing function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ an *approximation function*. For a given approximation function ψ , let

$$A_n(\psi) := \{\mathbf{x} \in \mathbb{R}^n : \exists^\infty \mathbf{q} \in \mathbb{Z}^n \text{ such that } \|\mathbf{q} \cdot \mathbf{x}\| < \psi(|\mathbf{q}|)\}. \quad (2)$$

Here \exists^∞ means “there exists infinitely many ...”.

Let $|\cdot|_{\mathcal{M}}$ denote the induced Lebesgue measure on \mathcal{M} . One wants to study the size of $A_n(\psi) \cap \mathcal{M}$ under $|\cdot|_{\mathcal{M}}$. Indeed, for $\mathcal{M} = \mathcal{V}_n$, A. Baker [3] conjectured that

Conjecture 2 (Baker).

$$|A_n(\psi) \cap \mathcal{M}|_{\mathcal{M}} = \begin{cases} \text{ZERO} & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q) < \infty \\ \text{FULL} & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q) = \infty. \end{cases}$$

Here $|\cdot|_{\mathcal{M}} = \text{FULL}$ means that the complement on \mathcal{M} has measure zero. The highly nontrivial special case \mathcal{V}_n was finally solved in [6, 17]. More importantly, it has been subsequently shown that the conjecture 2 holds for all non-degenerate submanifolds of \mathbb{R}^n [19, 9, 7]. So the Lebesgue measure theory for Groshev type approximation on manifolds is complete.

Beyond this, one may develop the much deeper Hausdorff theory in which one replaces the coarse Lebesgue measure $|\cdot|_{\mathcal{M}}$ with finer Hausdorff measures \mathcal{H}^s . We will recall the definitions of Hausdorff measure and dimension in §2. As a byproduct, this approach yields the Hausdorff dimension of the set in question. This type of investigation seems to be initiated by Baker and Schmidt [4] in 1970, who proved that

$$\frac{n+1}{\nu+1} \leq \dim(A_n(q^{-\nu}) \cap \mathcal{V}_n) \leq \frac{2(n+1)}{\nu+1} \quad \text{for any } \nu > n. \quad (3)$$

In the same paper, they also conjectured that the left inequality in (3) should be equality, which was eventually solved by Bernik [16] in 1983. In 2000, Dickinson and Dodson [20] proved that for any extremal submanifold \mathcal{M} of \mathbb{R}^n one has the lower bound

$$\dim(A_n(q^{-\nu}) \cap \mathcal{M}) \geq \frac{n+1}{\nu+1} + \dim \mathcal{M} - 1 \quad \text{for any } \nu > n. \quad (4)$$

Proving that the lower bound in (4) is actually sharp represents a major open question, which is called the Generalized Baker-Schmidt Problem for Hausdorff dimension in [2].

Conjecture 3 (GBSP for Hausdorff dimension). For any submanifold \mathcal{M} of \mathbb{R}^n which is non-degenerate everywhere except possibly on a set of Hausdorff dimension $\leq D$, one has

$$\dim(A_n(q^{-\nu}) \cap \mathcal{M}) = D \quad \text{for any } \nu > n$$

where

$$D := \frac{n+1}{\nu+1} + \dim \mathcal{M} - 1.$$

Besides the Veronese curve \mathcal{V}_n , Conjecture 3 was also known for non-degenerate planar curves due to R.C. Baker [5]. By setting $\psi(q) = q^{-\nu}$, we immediately see that Conjecture 3 is an easy corollary of the following very precise and delicate conjecture, which is referred as the Generalized Baker-Schmidt Problem for Hausdorff measure in [2].

Conjecture 4 (GBSP for Hausdorff measure). For any approximation function ψ , any $s > m - 1$ and any submanifold \mathcal{M} of \mathbb{R}^n of dimension m which is non-degenerate everywhere except possibly on a set of zero Hausdorff s -measure¹, one has

$$\mathcal{H}^s(A_n(\psi) \cap \mathcal{M}) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q} \right)^{s+1-m} q^n < \infty, \\ \mathcal{H}^s(\mathcal{M}) & \text{if } \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q} \right)^{s+1-m} q^n = \infty. \end{cases}$$

Clearly, the case $s = m$ reduces to the Lebesgue measure theory and hence was known. The case $s > m$ is simply trivial. When $s < m$, the divergence case of Conjecture 4 has been established by Beresnevich, Dickinson and Velani [10] using their general ubiquity framework. However, the convergence case is much more difficult and represents a challenging question. Indeed, it is not known for any non-degenerate manifolds except for the parabola which was solved very recently [24]. In this paper, we will settle this conjecture for all non-degenerate planar curves. This is the first result of its kind for a general class of manifolds.

¹In their original formulation of GBSP for Hausdorff dimension and measure, the authors of [2] seem to have overlooked this absolutely necessary condition.

Before stating our main theorem, we should also remark that there is a completely parallel line of developments in the context of simultaneous approximation on manifolds, in which one considers the set

$$S_n(\psi) := \{\mathbf{x} \in \mathbb{R}^n : \exists^\infty q \in \mathbb{Z} \text{ such that } \max_{1 \leq i \leq n} \|qx_i\| < \psi(q)\}, \quad (5)$$

instead of the set $A_n(\psi)$, and the measure-theoretic property of the intersection $S_n(\psi) \cap \mathcal{M}$. One can then formulate a complete analogue of Conjecture 4, namely the Hausdorff theory for simultaneous approximation on manifolds (see [8, Conjecture 8.2] for the precise statement). This analogue of Conjecture 4 for all non-degenerate planar curves has been remarkably established by Vaughan and Velani [32] for the convergence case, and Beresnevich, Dickinson and Velani [11] for the divergence case. Moreover, the divergence case of this analogous conjecture for all non-degenerate analytic manifolds has been subsequently solved by Beresnevich [8]. Despite the superficial similarity between the simultaneous and dual approximations on manifolds, the two problems exhibit quite differing natures and difficulties; hence the solution of one of them unfortunately does not yield that of the other one by any reasonable means. Our main theorem below is a natural counterpart of the result of Vaughan and Velani [32] in the dual setting, and hence brings the development of the dual approximation on manifolds in line with that of the simultaneous approximation on manifolds.

1.2. The main result.

Main Theorem. *Conjecture 4 holds when \mathcal{M} is a curve in \mathbb{R}^2 . Namely, for any approximation function ψ , $s \in (0, 1]$ and any C^2 planar curve \mathcal{C} which is non-degenerate everywhere except possibly on a set of zero Hausdorff s -measure, we have*

$$\mathcal{H}^s(A_2(\psi) \cap \mathcal{C}) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q}\right)^s q^2 < \infty, \\ \mathcal{H}^s(\mathcal{C}) & \text{if } \sum_{q=1}^{\infty} \left(\frac{\psi(q)}{q}\right)^s q^2 = \infty. \end{cases}$$

As remarked above, the divergence case in the Main Theorem is established in [10]. So the convergence case is the new substance, which we will prove in §3.

One can replace the Hausdorff s -measure \mathcal{H}^s in the Main Theorem with a Hausdorff g -measure \mathcal{H}^g for a general dimension function g , assuming some growth condition on g like in [24, Eqn. (5)]. However, this reveals no additional information on the problem and will inevitably

obscure the main thrust of the paper. We are happy to leave this to the reader.

1.3. Further general discussions. Our approach in this paper is partially motivated by the work of Schmidt [28, 29], who showed in 1964 that every C^3 non-degenerate curve is extremal. This result of Schmidt, together with Sprindžuk's solution of Mahler's conjecture [30], undoubtedly represent two pioneering work in the subject. In spite of this, the two methods they introduced differ quite significantly in nature. Sprindžuk invented the method which is now called *the method of essential and inessential domains* (see [30, 31]); while Schmidt's method is based on an interesting arithmetic counting problem [28] which is now understood to be closely related to the problem of counting rational points lying near planar curves. As discussed above, the Hausdorff theory is much deeper than the extremal theory and hence requires very strong arithmetic input, which Schmidt's result [28] unfortunately fails to deliver. The novel feature of this paper is that we adapt a counting result of the same type due to Huxley [25], which is essentially best possible, into an analytic form that is particularly suited for the application in Schmidt's method. This connection is established through the use of the *dual curve*, which appears, in one form or another, in the previous works [11, 15, 23, 25, 32].

Another interesting feature to be seen, on checking our argument carefully, is that we do not need the full power of Huxley [25]. Actually, we still obtain results of the same quality when the upper bound in Lemma 4 is relaxed to be $\ll \delta^\alpha R^\beta + R^{3/2-\varepsilon}$ for some $\varepsilon > 0$ and α, β satisfying $\beta < \frac{3}{2} + 2\alpha$. In contrast, we recall that when the corresponding Jarník type theorem for simultaneous approximation on planar curves was established in [32] one had to use Huxley in the strongest form. This discrepancy is probably not surprising since it is generally believed that dual approximation on manifolds is not more difficult than simultaneous approximation on manifolds. Nevertheless, it is indeed a bit surprising that the dual case for planar curves remained open for almost a decade after the simultaneous case was solved completely [11, 32]!

We emphasize that the relation between the distribution of rational points near manifolds and simultaneous approximation on manifolds is widely known and well understood [11, 32, 15, 8, 23]. However, to the best of our knowledge, the explicit relation between this arithmetic counting problem and dual approximation on manifolds has never appeared in the literature; hence it is the purpose of this paper to address this issue. Beyond Khintchine's transference principle, little has been

known about the relation between simultaneous and dual approximations. One may speculate that there are possibly deeper connections between them through arithmetic properties (for instance the distribution of rational points near manifolds).

We remark in passing that GBSP has been extended to the settings of inhomogeneous approximation and/or multi-variable approximation function. See [13, 5, 2] and references therein. The inhomogeneous and multiplicative theory have also been developed in the simultaneous case [14, 12]. It is likely that the method in this paper can be adapted in order to cope with the above variations of the original problem. We hope to return to this in a future publication. We note however that it is straightforward to see that one may insert an inhomogeneous constant term θ in our problem and the whole argument still works.

2. HAUSDORFF MEASURE AND DIMENSION

Here for completeness we recall the definitions of Hausdorff measure and dimension. For more examples and discussions, one is referred to the book of Falconer [21] and references therein. Let X be a subset of \mathbb{R}^n and $\{B_i\}$ be a collection of balls in \mathbb{R}^n with diameters $\text{diam}(B_i) < \rho$ for a fixed $\rho > 0$ such that $X \subseteq \bigcup_i B_i$, which is called a ρ -cover of X .

Now define

$$\mathcal{H}_\rho^s(X) := \inf \left\{ \sum_i (\text{diam}(B_i))^s \right\}$$

where the infimum is taken over all possible ρ -covers of X . The Hausdorff s -measure \mathcal{H}^s is defined as

$$\mathcal{H}^s(X) := \lim_{\rho \rightarrow 0} \mathcal{H}_\rho^s(X).$$

Due to monotonicity, this limit either converges to some finite number or approaches ∞ .

When s is a positive integer, \mathcal{H}^s is equivalent to the usual s -dimensional Lebesgue measure.

Moreover, the *Hausdorff dimension* $\dim X$ of X is defined by

$$\dim X := \inf\{s : \mathcal{H}^s(X) = 0\} = \sup\{s : \mathcal{H}^s = \infty\}.$$

Namely the Hausdorff dimension of X is the unique point s where the Hausdorff s -measure jumps from ∞ to 0. It is particularly useful when analyzing the sizes of fractals. Though we do not attempt to make this precise, we will remark in passing that the set $A_n(\psi)$ in our question behaves in many aspects like a fractal, and in particular is self-similar.

We conclude this section by stating the *Hausdorff-Cantelli Lemma*², which will be used repeatedly in §3.

Lemma 1 (Hausdorff-Cantelli). *Let H_i be a sequence of hypercubes in \mathbb{R}^n and suppose that for some $s > 0$,*

$$\sum_i (|H_i|)^s < \infty,$$

then $\mathcal{H}^s(\limsup H_i) = 0$.

Here $|H_i|$ is the n -dimensional Lebesgue measure of H_i and the lim-sup set

$$\limsup H_i := \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} H_j$$

contains exactly those elements that belong to infinitely many of H_i .

3. PROOF OF THE MAIN THEOREM

In this section, we embark on the proof of the Main Theorem. For convenience we make some standard simplifications.

By changing coordinates, a C^2 planar curve \mathcal{C} is locally represented by graphs $\mathcal{C}_f := \{(x, f(x)) : x \in I\}$ of C^2 functions f on some compact interval I .

Clearly, the local coordinate map $\mathbf{f} : I \rightarrow \mathcal{C}_f$ which sends x to $(x, f(x))$ is bi-Lipschitz and hence for any set $\mathcal{A} \in I$ we have

$$\mathcal{H}^s(\mathcal{A}) \asymp \mathcal{H}^s(\mathbf{f}(\mathcal{A})).$$

Let $B := \{x \in I : f''(x) = 0\}$. Recall that non-degeneracy at a point x simply means that $f''(x) \neq 0$. Since \mathcal{C} is non-degenerate everywhere except on a set of zero Hausdorff s -measure, we have $\mathcal{H}^s(B) = 0$. By continuity, B is a closed set in I , hence a simple measure-theoretic argument allows us to write $I \setminus B$ as a countable union of bounded open intervals I_i on which f satisfies

$$0 < c_1 \leq |f''(x)| \leq c_2 < \infty. \tag{6}$$

for all $x \in I_i$. Here the positive constants c_1 and c_2 depend on the particular choice of interval I_i . Since the Hausdorff measure is countably subadditive, to prove the Main Theorem for all C^2 non-degenerate planar curves, it suffices to prove it for all arcs $\mathcal{C}_f := \{(x, f(x)) : x \in I\}$ associated with some C^2 function f whose second derivative is bounded

²This is referred as the Hausdorff-Cantelli lemma in [18] for convenience and should not be taken as a joint work.

and bounded away from zero. Thus, without loss of generality, we will assume f satisfies (6) on I .

Let $\psi(q) : \mathbb{N} \rightarrow \mathbb{R}^+$ be an approximation function. Write $I := [a, b]$. For $(q_1, q_2, p) \in \mathbb{Z}^3$, let

$$\mu(q_1, q_2, p) = \{x \in I : |q_1 x + q_2 f(x) - p| < \psi(q)\} \quad (7)$$

where $q = \max\{|q_1|, |q_2|\}$. We are primarily interested in the measure theoretic properties of the set

$$\mu(\psi) = \{x \in I : \exists^\infty (q_1, q_2, p) \in \mathbb{Z}^3 \text{ such that } x \in \mu(q_1, q_2, p)\}. \quad (8)$$

Notice that $\mathcal{H}^s(\mu(\psi)) = 0$ if and only if $\mathcal{H}^s(A_2(\psi) \cap \mathcal{C}_f) = 0$. Thus the convergence case of the Main Theorem reduces to

Proposition. *Let $s \in (0, 1]$. Then*

$$\mathcal{H}^s(\mu(\psi)) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} \psi(q)^s q^{2-s} \text{ converges.}$$

First of all, due to the convexity assumption (6), we observe the fact that $\mu(q_1, q_2, p)$ is either an interval or the union of two intervals. Hence in view of the Hausdorff-Cantelli Lemma, to prove the proposition it suffices to show the convergence of the series

$$\sum_{\substack{q_1, q_2, p \in \mathbb{Z} \\ (q_1, q_2) \neq (0, 0)}} |\mu(q_1, q_2, p)|^s \quad (9)$$

provided that

$$\sum_{q=1}^{\infty} \psi(q)^s q^{2-s} < \infty. \quad (10)$$

We may assume, by a standard reduction argument,

$$\psi(q) \geq q^{1-\frac{3+\varepsilon_0}{s}} \quad (11)$$

with some fixed small $\varepsilon_0 > 0$. Because if ψ does not satisfy (11), we may take a new approximation function

$$\hat{\psi}(q) := \max \left\{ \psi(q), q^{1-\frac{3+\varepsilon_0}{s}} \right\},$$

which satisfies (11). One can easily check that

$$\sum_{q=1}^{\infty} \hat{\psi}(q)^s q^{2-s} < \infty.$$

We may define the corresponding set $\hat{\mu}(q_1, q_2, p)$ as in (7) with ψ replaced by $\hat{\psi}$. Clearly the inclusion relation

$$\mu(q_1, q_2, p) \subseteq \hat{\mu}(q_1, q_2, p)$$

holds since $\hat{\psi}(q) \geq \psi(q)$. So if we can show $\mathcal{H}^s(\mu(\hat{\psi})) = 0$, then $\mathcal{H}^s(\mu(\psi)) = 0$ follows. Hence without loss of generality, for the rest of the argument, we will assume (11).

We will divide the set $\mathbb{Z}^2 \setminus (0, 0)$ of all possible choices for (q_1, q_2) into a couple of subsets, which will be dealt with separately. We also notice that for given (q_1, q_2) , there are only finitely many $p \in \mathbb{Z}$ such that $\mu(q_1, q_2, p) \neq \emptyset$. Indeed, in view of the definition (7) such p must satisfy

$$|p| \leq Cq = C \max\{|q_1|, |q_2|\} \quad (12)$$

where

$$C = \max_{x \in I} \{|x| + |f(x)| + 1\}.$$

Let

$$M = 1 + \max_{x \in I} |f'(x)|$$

and

$$\Theta_1 = \{(q_1, q_2) \in \mathbb{Z} \times \mathbb{Z} : |q_1| > 2M|q_2|\}$$

and

$$\Theta_2 = \mathbb{Z}^2 \setminus \Theta_1 \cup (0, 0).$$

We state a lemma (see [22, Lemma 9.7]) that will be used repeatedly in our argument.

Lemma 2. *Let $h(x) \in C^2(I)$ be such that $\min_{x \in I} |h'(x)| = \delta_1$ and $\min_{x \in I} |h''(x)| = \delta_2$. For $\eta > 0$, define*

$$E(\eta) := \{\eta \in I : |h(x)| < \eta\}.$$

Then we have

$$|E(\eta)| \ll \min \left(\frac{\eta}{\delta_1}, \sqrt{\frac{\eta}{\delta_2}} \right).$$

We consider the case $(q_1, q_2) \in \Theta_1$ first. In this case, we have

$$|q_1 + q_2 f'(x)| \geq |q_1| - M|q_2| \geq \frac{|q_1|}{2}.$$

Now by Lemma 2, we know

$$|\mu(q_1, q_2, p)| \ll \frac{\psi(|q_1|)}{|q_1|}.$$

Moreover, for given q_1 , there are at most $\ll q_1^2$ possible choices for q_2 and p . Therefore the total contribution from the $(q_1, q_2) \in \Theta_1$ to (9) is

$$\begin{aligned} & \sum_{\substack{(q_1, q_2) \in \Theta_1 \\ p \in \mathbb{Z}}} |\mu(q_1, q_2, p)|^s \\ & \ll \sum_{q_1 \in \mathbb{Z}^*} (\psi(|q_1|)/|q_1|)^s |q_1|^2 \end{aligned}$$

which is convergent by (10).

From now on, we treat the much more difficult case $(q_1, q_2) \in \Theta_2$. Clearly in this case $q_2 \neq 0$ and moreover $|q_2| \asymp q$. For convenience, we may extend the definition of $f(x)$ to \mathbb{R} by taking the second order Taylor expansions at the end points of I . Namely let

$$f(x) = f(b) + f'(b)(x - b) + \frac{f''(b)}{2}(x - b)^2$$

when $x > b$ and

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

when $x < a$. Clearly the extended f satisfies $f \in C^2(\mathbb{R})$ and $c_1 \leq |f''| \leq c_2$. Since f'' does not change sign throughout \mathbb{R} , f' is strictly monotonic on \mathbb{R} and has range $(-\infty, \infty)$. Let $g(y) : \mathbb{R} \rightarrow \mathbb{R}$ be the inverse function of $-f'(x)$. To this end, let

$$x_0 := g(q_1/q_2)$$

which is the unique point $x_0 \in \mathbb{R}$ such that

$$q_1 + q_2 f'(x_0) = 0. \quad (13)$$

Let $J = [-2M, 2M]$ and $I' = g(J) \supset I$. So $x_0 \in I'$. Note that

$$g'(y) = \frac{1}{f''(g(y))}$$

and hence that

$$c_2^{-1} \leq |g'(y)| \leq c_1^{-1} \quad (14)$$

for all $y \in \mathbb{R}$. Thus by the mean value theorem

$$|I'| \leq c_1^{-1} |J| = 4c_1^{-1} M. \quad (15)$$

Now, we let

$$F(x) = q_1 x + q_2 f(x)$$

with $q_1/q_2 \in J$. Then

Lemma 3.

$$|F'(x)| \asymp |q_2(F(x) - F(x_0))|^{1/2}$$

where the \asymp constants depend only on c_1, c_2 .

Proof. Applying Taylor's theorem of order 2 about $x = x_0$ for $F(x)$ and noting $F'(x_0) = 0$ by (13), we obtain

$$F(x) = F(x_0) + \frac{F''(\xi_1)}{2}(x - x_0)^2 \quad (16)$$

for some ξ_1 between x and x_0 .

Moreover, by the mean value theorem,

$$F'(x) = F'(x) - F'(x_0) = F''(\xi_2)(x - x_0) \quad (17)$$

for some ξ_2 between x and x_0 .

Hence merging (16) and (17) yields

$$F(x) - F(x_0) = \frac{F''(\xi_1)}{2} \left(\frac{F'(x)}{F''(\xi_2)} \right)^2.$$

The lemma immediately follows on noticing that

$$c_1|q_2| \leq |F''(x)| \leq c_2|q_2|.$$

□

Now let p_0 be the unique integer such that

$$-\frac{1}{2} < F(x_0) - p_0 \leq \frac{1}{2}. \quad (18)$$

If $p \neq p_0$, then for $x \in \mu(q_1, q_2, p)$

$$\begin{aligned} |F(x) - F(x_0)| &= |p - p_0 + F(x) - p - F(x_0) + p_0| \\ &\geq |p - p_0| - |F(x) - p| - |F(x_0) - p_0| \\ &\stackrel{(18)}{\geq} |p - p_0| - \psi(q) - 1/2 \\ &\geq \frac{1}{3}|p - p_0| \end{aligned}$$

provided that

$$\psi(q) \leq \frac{1}{8}. \quad (19)$$

Since $\psi(q)$ decreases monotonically to 0 as $q \rightarrow \infty$, there exists $q_0 \in \mathbb{N}$ such that (19) holds when $q \geq q_0$.

By Lemma 3 and then Lemma 2, we get, if $p \neq p_0$ and $q \geq q_0$ then

$$|\mu(q_1, q_2, p)| \ll \psi(q)(|q_2||p - p_0|)^{-1/2}. \quad (20)$$

Therefore for fixed $(q_1, q_2) \in \Theta_2$ with $\max\{|q_1|, |q_2|\} \geq q_0$

$$\begin{aligned} \sum_{p \neq p_0} |\mu(q_1, q_2, p)|^s &\ll \sum_{p \neq p_0} \psi(q)^s (q|p - p_0|)^{-s/2} \\ &\stackrel{(12)}{\ll} \psi(q)^s q^{-s/2} q^{1-s/2} \\ &\ll \psi(q)^s q^{1-s}. \end{aligned}$$

For fixed q , there are at most $\ll q$ choices for q_1, q_2 . Hence

$$\sum_{\substack{(q_1, q_2) \in \Theta_2 \\ \max\{|q_1|, |q_2|\} \geq q_0 \\ p \neq p_0}} |\mu(q_1, q_2, p)|^s \ll \sum_q \psi(q)^s q^{2-s}$$

which converges.

Hitherto, we are left with the most difficult case $p = p_0$. For $x \in \mu(q_1, q_2, p_0)$, we have

$$\begin{aligned} |F(x) - F(x_0)| &= |F(x) - p_0 + p_0 - F(x_0)| \\ &\geq \|F(x_0)\| - \psi(q) \\ &\geq \frac{1}{2} \|F(x_0)\| \end{aligned}$$

provided that

$$\|F(x_0)\| \geq 2\psi(q).$$

By Lemma 3

$$|F'(x)| \gg (q\|F(x_0)\|)^{1/2}.$$

Then by Lemma 2

$$|\mu(q_1, q_2, p_0)| \ll \frac{\psi(q)}{q^{1/2}} \|F(x_0)\|^{-1/2}. \quad (21)$$

On the other hand, if $\|F(x_0)\| < 2\psi(q)$, then

$$|\mu(q_1, q_2, p_0)| \stackrel{\text{Lem. 2}}{\ll} \sqrt{\frac{\psi(q)}{q}}. \quad (22)$$

We now define the dual curve $f^*(y)$ of $f(x)$, whose derivative is the inverse function of $-f'(x)$. Namely

$$f^*(y) := yg(y) + f(g(y)).$$

It is readily seen that

$$(f^*)'(y) = g(y) + yg'(y) + f'(g(y))g'(y) = g(y) \quad (23)$$

and that

$$q_2 f^*(q_1/q_2) = q_1 g(q_1/q_2) + q_2 f(g(q_1/q_2)) = q_1 x_0 + q_2 f(x_0) = F(x_0). \quad (24)$$

The following lemma, which can be found as Lemma 2.2 in [32], is essentially due to Huxley [25].

Lemma 4 (Huxley). *Suppose that ϕ has a continuous second derivative on a bounded interval Γ which is bounded away from 0, and let $\delta \in (0, \frac{1}{4})$. Then for any $\varepsilon > 0$ and $R \geq 1$,*

$$\sum_{R \leq r < 2R} \sum_{\substack{t/r \in \Gamma \\ \|r\phi(t/r)\| < \delta}} 1 \ll_{\varepsilon} \delta^{1-\varepsilon} R^2 + R \log(2R).$$

Lemma 5 is a consequence of Lemma 4.

Lemma 5. *Under the same conditions with Lemma 4, for $\lambda \in (0, 1)$ and $R \geq 1$,*

$$\sum_{R \leq r < 2R} \sum_{\substack{t/r \in \Gamma \\ \|r\phi(t/r)\| \geq \delta}} \left\| r\phi\left(\frac{t}{r}\right) \right\|^{-\lambda} \ll R^2 + \delta^{-\lambda} R \log(2R).$$

Proof. We can restrict our attention to the terms with $\|r\phi(t/r)\| < 1/4$ since the terms with $\|r\phi(t/r)\| \geq 1/4$ clearly contribute $\ll R^2$ in total to the sum. We separate the values of $\|r\phi(t/r)\|$ into dyadic ranges and obtain

$$\begin{aligned} \sum_{R \leq r < 2R} \sum_{\substack{t/r \in \Gamma \\ 1/4 > \|r\phi(t/r)\| > \delta}} \left\| r\phi\left(\frac{t}{r}\right) \right\|^{-\lambda} &\ll \sum_{j \leq \log_2 \frac{1}{4\delta}} \sum_{R \leq r < 2R} \sum_{\substack{t/r \in \Gamma \\ \|r\phi(t/r)\| < 2^j \delta}} (2^{j-1} \delta)^{-\lambda} \\ &\stackrel{\text{Lem. 4}}{\ll} \sum_j 2^{-j\lambda} \delta^{-\lambda} ((2^j \delta)^{1-\varepsilon} R^2 + R \log(2R)) \\ &\ll 2^{(1-\lambda-\varepsilon) \log_2 \frac{1}{4\delta}} \delta^{1-\lambda-\varepsilon} R^2 + \delta^{-\lambda} R \log(2R) \\ &\ll R^2 + \delta^{-\lambda} R \log(2R). \end{aligned}$$

Here we choose $\varepsilon = \frac{1-\lambda}{2}$ when applying Lemma 4. \square

Finally, we are poised to prove that the series

$$\sum_{(q_1, q_2) \in \Theta_2} |\mu(q_1, q_2, p_0)|^s < \infty \quad (25)$$

and hence conclude the proof of the proposition. We further divide this into two cases, namely $\|F(x_0)\| < 2\psi(q)$ and $\|F(x_0)\| \geq 2\psi(q)$.

For any integer k with $2^k \geq q_0$,

$$\begin{aligned}
 & \sum_{\substack{(q_1, q_2) \in \Theta_2 \\ 2^k \leq |q_2| < 2^{k+1} \\ \|F(x_0)\| < 2\psi(q)}} |\mu(q_1, q_2, p_0)|^s \\
 & \stackrel{(24) \& (22)}{\ll} \sum_{2^k \leq q_2 < 2^{k+1}} \sum_{\substack{q_1/q_2 \in J \\ \|q_2 f^*(q_1/q_2)\| < 2\psi(2^k)}} \left(\frac{\psi(2^k)}{2^k} \right)^{\frac{s}{2}} \\
 & \stackrel{\text{Lem. 4}}{\ll} (\psi(2^k)^{1-\varepsilon} 2^{2k} + k 2^k) \left(\frac{\psi(2^k)}{2^k} \right)^{\frac{s}{2}} \\
 & \ll \psi(2^k)^{1+\frac{s}{2}-\varepsilon} 2^{(2-s/2)k} + k \psi(2^k)^{\frac{s}{2}} 2^{(1-s/2)k}.
 \end{aligned}$$

This is

$$\ll \psi(2^k)^s (2^k)^{3-s}$$

provided that

$$\psi(2^k) \geq (2^k)^{1-\frac{4-\varepsilon_1}{s}} \quad (26)$$

for some small $\varepsilon_1 > 0$. Now recalling (11), the condition (26) is always satisfied! Therefore

$$\sum_{\substack{(q_1, q_2) \in \Theta_2 \\ |q_2| \geq q_0 \\ \|F(x_0)\| < 2\psi(q)}} |\mu(q_1, q_2, p_0)|^s \ll \sum_k \psi(2^k)^s (2^k)^{3-s} \stackrel{(10)}{<} \infty. \quad (27)$$

The other case can be treated in a similar fashion.

$$\begin{aligned}
 & \sum_{\substack{(q_1, q_2) \in \Theta_2 \\ 2^k \leq |q_2| < 2^{k+1} \\ \|F(x_0)\| \geq 2\psi(q)}} |\mu(q_1, q_2, p_0)|^s \\
 & \stackrel{(24) \& (21)}{\ll} \sum_{2^k \leq q_2 < 2^{k+1}} \sum_{\substack{q_1/q_2 \in J \\ \|q_2 f^*(q_1/q_2)\| \geq 2\psi(2^k)}} \left(\frac{\psi(2^k)}{2^{k/2}} \right)^s \|q_2 f^*(q_1/q_2)\|^{-s/2} \\
 & \stackrel{\text{Lem. 5}}{\ll} (\psi(2^k)^{-s/2} 2^k k + 2^{2k}) \left(\frac{\psi(2^k)}{2^{k/2}} \right)^s \\
 & \ll \psi(2^k)^{\frac{s}{2}} (2^k)^{1-s/2} k + \psi(2^k)^s (2^k)^{(2-s/2)}.
 \end{aligned}$$

Again this is

$$\ll \psi(2^k)^s (2^k)^{3-s}$$

provided that (26) holds, which is guaranteed by (11). So

$$\sum_{\substack{(q_1, q_2) \in \Theta_2 \\ |q_2| \geq q_0 \\ \|F(x_0)\| \geq 2\psi(q)}} |\mu(q_1, q_2, p_0)|^s \ll \sum_k \psi(2^k)^s (2^k)^{3-s} \stackrel{(10)}{<} \infty. \quad (28)$$

Now, (25) follows from (27) and (28). Then the proposition follows from (9) and the Hausdorff-Cantelli lemma.

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REFERENCES

- [1] D. Badziahin, *Inhomogeneous Diophantine approximation on curves and Hausdorff dimension*. Adv. Math. 223 (2010) 329351.
- [2] D. Badziahin; V. Beresnevich; S. Velani, *Inhomogeneous theory of dual Diophantine approximation on manifolds*. Adv. Math. 232 (2013), 1-35.
- [3] A. Baker, *Transcendental Number Theory*. Cambridge University Press, London, 1975.
- [4] A. Baker, W.M. Schmidt, *Diophantine approximation and Hausdorff dimension*. Proc. Lond. Math. Soc. 21 (1970) 1-11.
- [5] R.C. Baker, *Dirichlets theorem on Diophantine approximation*. Math. Proc. Cam. Phil. Soc. 83 (1978) 37-59.
- [6] V. Beresnevich, *On approximation of real numbers by real algebraic numbers*. Acta Arith. 99 (1999) 97-112.
- [7] V. Beresnevich, *A Groshev type theorem for convergence on manifolds*. Acta Math. Hungar. 94 (2002) 99-130.
- [8] V. Beresnevich, *Rational points near manifolds and metric Diophantine approximation*. Ann. of Math. 175 (2) (2012) 187-235.
- [9] V. Beresnevich; V. Bernik; D. Kleinbock; G.A. Margulis, *Metric Diophantine approximation: the Khintchine- Groshev theorem for nondegenerate manifolds*. Mosc. Math. J. 2 (2002) 203-225.
- [10] V. Beresnevich; D. Dickinson; S. Velani, *Measure theoretic laws for lim sup sets*. Mem. Amer. Math. Soc. 179 (2006) x+91.
- [11] V. Beresnevich; D. Dickinson; S. Velani, *Diophantine approximation on planar curves and the distribution of rational points*. Ann. of Math. 166 (2) (2007) 367-426. With an Appendix II by R.C. Vaughan.
- [12] V. Beresnevich; R.C. Vaughan; S. Velani, *Inhomogeneous Diophantine approximation on planar curves*. Math. Ann. 349 (2011) 929-942.
- [13] V. Beresnevich; S. Velani, *An inhomogeneous transference principle and Diophantine approximation*. Proc. Lond. Math. Soc. 101 (2010) 821-851.
- [14] V.V. Beresnevich; S.L. Velani, *A note on simultaneous Diophantine approximation on planar curves*. Math. Ann. 337 (2007), no. 4, 769-796.
- [15] V. Beresnevich; E. Zorin, *Explicit bounds for rational points near planar curves and metric Diophantine approximation*. Adv. Math. 225 (2010) 3064-3087.

- [16] V. Bernik, *An application of Hausdorff dimension in the theory of Diophantine approximation*. Acta Arith. 42 (1983) 219-253 (in Russian) English transl. in Amer. Math. Soc. Transl. 140 (1988) 15-44.
- [17] V.I. Bernik, *On the exact order of approximation of zero by values of integral polynomials*. Acta Arith. 53 (1989) 17-28.
- [18] V. Bernik; M. Dodson, *Metric Diophantine approximation on manifolds*. Cambridge Tracts in Mathematics, vol. 137, Cambridge University Press, Cambridge, 1999.
- [19] V. Bernik; D. Kleinbock; G.A. Margulis, *Khintchine-type theorems on manifolds: the convergence case for standard and multiplicative versions*. Internat. Math. Res. Not. (2001) 453-486.
- [20] H. Dickinson; M. Dodson, *Extremal manifolds and Hausdorff dimension*. Duke Math. J. 101 (2000) 271-281.
- [21] K. Falconer, *Fractal geometry*. Mathematical foundations and applications. Second edition. John Wiley & Sons, Inc., Hoboken, NJ, 2003.
- [22] G. Harman, *Metric number theory*. London Mathematical Society Monographs. New Series, 18. The Clarendon Press, Oxford University Press, New York, 1998. xviii+297 pp.
- [23] J.-J. Huang, *Rational points near planar curves and Diophantine approximation*. Preprint, arXiv:1403.7388 [math.NT].
- [24] M. Hussain, *A Jarnik type theorem for planar curves: everything about the parabola*. Preprint, arXiv:1309.7112 [math.NT].
- [25] M.N. Huxley, *The rational points close to a curve*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 21 (1994), no. 3, 357-375.
- [26] D. Kleinbock; G.A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*. Ann. of Math. (2) 148 (1998) 339-360.
- [27] K. Mahler, *Über das Maß der Menge aller S -Zahlen*. Math. Ann. 106 (1932) 131-139.
- [28] W.M. Schmidt, *Über Gitterpunkte auf gewissen Flächen*. (German) Monatsh. Math. 68 1964 59-74.
- [29] W.M. Schmidt, *Metrische Sätze über simultane Approximation abhängiger Größen*. (German) Monatsh. Math. 68 1964 154-166.
- [30] V.G. Sprindžuk, *Mahler's Problem in Metric Number Theory*. Translations of Mathematical Monographs, vol. 25, American Mathematical Society, Providence, R.I, 1969.
- [31] V.G. Sprindžuk, *Metric theory of Diophantine approximations*. John Wiley & Sons, 1979.
- [32] R.C. Vaughan; S. Velani, *Diophantine approximation on planar curves: the convergence theory*. Invent. Math. 166 (2006), no. 1, 103-124.

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